1 Problem 1

1. Evaluate the integral: $\int_0^{\pi/2} \sin^2(t)\cos^5(t)\, dt$.

Well, split off a $\cos(t)$ since they have an odd power: $\int_0^{\pi/2} \sin^2(t)\cos^4(t)\cos(t)\, dt$. Then set $u = \sin(t)$ so $du = \cos(t)\, dt$, and turn $\cos^4(t) = (\cos^2 t)^2 = (1-\sin^2 t)^2$ since $\cos^2 t + \sin^2 t = 1$.

Now,

$$\begin{aligned}
\int_0^{\pi/2} \sin^2(t)\cos^4(t)\cos(t)\, dt = \int_0^{\pi/2} \sin^2(t)(1-\sin^2(t))^2\cos(t)\, dt &= \int_{u=\cos(0)}^{u=\cos(\pi/2)} u^2(1-u^2)^2\, du && (1)\\
&= \int_{u=1}^{u=0} u^2(1-2u^2+u^4)\, du && (2)\\
&= \int_1^0 u^6 - 2u^4 + u^2\, du && (3)\\
&= [u^7/7 - 2u^5/5 + u^3/3]_1^0 && (4)\\
&= 0 - (1/7 - 2/5 + 1/3) && (5)\\
&= \frac{2}{5} - \frac{1}{3} - \frac{1}{7} && (6)\\
&= \frac{2\cdot3\cdot7}{5\cdot3\cdot7} - \frac{5\cdot7}{3\cdot5\cdot7} - \frac{3\cdot5}{7\cdot3\cdot5} && (7)\\
&= \frac{42-35-15}{105} = \frac{-8}{105} && (8)
\end{aligned}$$

2 Problem 2

2. Evaluate the integral: $\int (1+\sin^2 x)(1+\cos^2 x)\, dx$.

First expand: $\int 1 + \sin^2 x + \cos^2 x + \sin^2 x\cos^2 x\, dx$. We can find $\int \sin^2 x\, dx$ and $\int \cos^2 x\, dx$ using the identities $\sin^2 x = (1-\cos 2x)/2$, $\cos^2 x = (1+\cos 2x)/2$:

$$\begin{aligned}
&\int \sin^2 x\, dx = \int (1-\cos 2x)/2\, dx = x/2 - (\sin 2x)/4 + C\\
&\int \cos^2 x\, dx = \int (1+\cos 2x)/2\, dx = x/2 + (\sin 2x)/4 + C
\end{aligned}$$

Thus we only need $\int \sin^2 x\cos^2 x\, dx$. Again, we use the identities $\sin^2 x = (1-\cos 2x)/2$, $\cos^2 x = (1+\cos 2x)/2$:

$\int \sin^2 x\cos^2 x\, dx = \int (1-\cos 2x)/2\cdot(1+\cos 2x)/2\, dx = \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)\, dx = \frac{1}{4}\int 1-\cos^2(2x)\, dx$

AGAIN, we use the identities: $\cos^2(2x) = (1-\cos(2(2x))/2 = (1-\cos(4x))/2$.

Then,

$\frac{1}{4}\int 1-\cos^2(2x)\, dx = \frac{1}{4}\int 1-(1-\cos(4x))/2\, dx = \frac{1}{4}\int 1/2 + \cos(4x)/2\, dx = \frac{1}{8}(x + \sin(4x)/4) + C$

Lastly add all we've found:

$$\begin{aligned}
\int (1+\sin^2 x)(1+\cos^2 x)\, dx &= \int 1 + \sin^2 x + \cos^2 x + \sin^2 x\cos^2 x\, dx && (9)\\
&= [x] + [x/2 - (\sin 2x)/4] + [x/2 + (\sin 2x)/4] + [\frac{1}{8}(x + \sin(4x)/4)] + C && (10)\\
&= 17x/8 + \sin(4x)/32 + C && (11)
\end{aligned}$$

3 Problem 3

3. Evaluate the integral: $\int \csc^3 t\, dt$.

We will proceed via integration by parts, letting $u = \csc t$, $dv = \csc^2 t\, dt$, so $du = -\csc t \cot t\, dt$, $v = -\cot t$. Then,

$$\int \csc^3 t\, dt = (\csc t)(-\cot t) - \int -\cot t \cdot -\csc t \cot t\, dt = -\csc t \cot t - \int \cot^2 t \csc t\, dt \tag{12}$$

We will use the equality $\cot^2 t + 1 = \csc^2 t$ (divide $\cos^2 t + \sin^2 t = 1$ by $\sin^2 t$), so

$$\int \cot^2(t)\csc(t)\, dt = \int (\csc^2 t - 1)\csc(t)\, dt = \int \csc^3(t) - \csc t\, dt.$$

Then,

$$\int \csc^3 t\, dt = -\csc t \cot t - \int \cot^2 t \csc t\, dt = -\csc t \cot t - (\int \csc^3 t - \csc t\, dt) \tag{13}$$

$$= -\csc t \cot t - \int \csc^3 t\, dt + \int \csc t\, dt \tag{14}$$

Now add $\int \csc^3 t\, dt$ to both sides, and note $\int \csc t\, dt = -\ln|\csc t + \cot t| + C$, so

$$2\int \csc^3 t\, dt = -\csc t \cot t - \ln|\csc t + \cot t| + C$$
$$\implies \int \csc^3 t\, dt = -\tfrac{1}{2}\csc t \cot t - \tfrac{1}{2}\ln|\csc t + \cot t| + C$$

4 Problem 4

4. Evaluate the integral: $\int_0^{\pi/4} \tan^5 t \sec^4 t\, dt$.

Since there's an odd number of $\tan t$, split off $\sec(t)\tan(t)$ to set up $u = \sec t$:

$$\int_0^{\pi/4} \tan^5 t \sec^4 t\, dt = \int_0^{\pi/4} \tan^4 t \sec^3 t(\sec t \tan t)\, dt$$

Change $\tan^4 t$ into $(\tan^2 t)^2 = (\sec^2 -1)^2$, then do our U-sub, $u = \sec t$, $du = \sec t \tan t\, dt$:

$$\int_0^{\pi/4} \tan^4 t \sec^3 t(\sec t \tan t)\, dt = \int_0^{\pi/4} (\sec^2 -1)^2 \sec^3 t(\sec t \tan t)\, dt \tag{15}$$

$$= \int_{u=\sec(0)}^{u=\sec(\pi/4)} (u^2-1)^2 u^3\, du \tag{16}$$

$$= \int_{u=1}^{u=\sqrt{2}/1} (u^4 - 2u^2 + 1)u^3\, du \tag{17}$$

$$= \int_{u=1}^{u=\sqrt{2}} (u^7 - 2u^5 + u^3)\, du \tag{18}$$

$$= [u^8 - u^6/3 + u^4/4]_1^{\sqrt{2}} \tag{19}$$

$$= (\sqrt{2}^8 - \sqrt{2}^6/3 + \sqrt{2}^4/4) - (1/8 - 1/3 + 1/4) \tag{20}$$

$$= (16 - 8/3 + 1) - (1/8 - 1/3 + 1/4) \tag{21}$$

$$= 17 - 7/3 - 3/8 = 17 - \frac{(56+21)}{24} \tag{22}$$

$$= 17 - 77/24 \tag{23}$$

$$= 485/24 \tag{24}$$

5 Problem 5

5. Evaluate the integral: $\int \frac{\tan x}{\sec^2 x}\,dx$.

Get everything in terms of $\sin x$, $\cos x$: $\int \frac{\tan x}{\sec^2 x}\,dx = \int \frac{\sin x}{\cos x} / \frac{1}{\cos^2 x}\,dx = \int \frac{\sin x \cos^2 x}{\cos x}\,dx = \int \sin x \cos x\,dx$. Now set $u = \sin x$, $du = \cos x\,dx$ so the integral is $\int u\,du = u^2/2 + C = \sin^2 x/2 + C$.

6 Problem 6

6. Evaluate the integral: $\int \sin 2x \sin 3x\,dx$.

We'll use the multiplication formula $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$ (remember where the minus signs go in the formula by thinking sinus=minus).

Then $\sin 2x \sin 3x = \frac{1}{2}(\cos(2x-3x) - \cos(2x+3x)) = \frac{1}{2}(\cos(-x) - \cos(5x)) = \frac{1}{2}(\cos x - \cos 5x)$, so $\int \sin 2x \sin 3x\,dx = \int \frac{1}{2}(\cos x - \cos 5x)\,dx = \frac{1}{2}(\sin x - (\sin 5x)/5) + C$

7 Problem 7

7. Evaluate the integral: $\int \sin^6 w \cos^4 w\,dw$.

Again, we use the identities $\sin^2 w = (1-\cos(2w))/2$, $\cos^2 w = (1+\cos(2w))/2$, turning $\sin^6 w = (\sin^2 w)^3$, $\cos^4 w = (\cos^2 w)^2$. Then

$$
\begin{aligned}
I = \int \sin^6 w \cos^4 w\,dw &= \int ((1-\cos 2w)/2)^3 \cdot ((1+\cos 2w)/2)^2\,dw && (25)\\
&= \frac{1}{2^5}\int (1-\cos 2w)^3(1+\cos 2w)^2\,dw && (26)\\
&= \frac{1}{2^5}\int (1-\cos 2w)[(1-\cos 2w)(1+\cos 2w)]^2\,dw && (27)\\
&= \frac{1}{2^5}\int (1-\cos 2x)[1-\cos^2(2w)]^2\,dw && (28)\\
& && (29)
\end{aligned}
$$

Here, we'll use our identity $\cos^2(2w) = (1+\cos(2(2w))/2 = (1+\cos 4w/2)$:

$$
\begin{aligned}
I = \frac{1}{2^5}\int (1-\cos 2x)[1-\cos^2(2w)]^2\,dw &= \frac{1}{32}\int (1-\cos 2x)[1-(1+\cos(4w))/2]^2\,dw && (30)\\
&= \frac{1}{2^5}\int (1-\cos 2x)[1/2-\cos(4w))/2]^2\,dw && (31)\\
&= \frac{1}{2^5}\int \frac{1}{2^2}(1-\cos 2w)(1-\cos 4w)^2\,dw && (32)\\
&= \frac{1}{2^7}\int (1-\cos 2w)(1-2\cos 4w+\cos^2 4w)\,dw && (33)\\
& && (34)
\end{aligned}
$$

Here' we'll use the identity $\cos^2(4w) = (1 + \cos(2(4w)))/2 = (1 + \cos 8w)/2$:

$$\frac{1}{2^7}\int(1-\cos 2w)(1-2\cos 4w+\cos^2 4w)\,dw = \frac{1}{2^7}\int(1-\cos 2w)(1-2\cos 4w+(1+\cos 8w)/2)\,dw \tag{35}$$

$$= \frac{1}{2^7}\int(1-\cos 2w)(3/2-2\cos 4w+\cos 8w/2)\,dw \tag{36}$$

$$= \frac{1}{2^8}\int(1-\cos 2w)(3-4\cos 4w+\cos 8w)\,dw \tag{37}$$

We expand (37):

$$I = \frac{1}{2^8}\int 3-4\cos 4w+\cos 8w-3\cos 2w+4\cos 4w\cos 2w-\cos 8w\cos 2w\,dw \tag{38}$$

$$= \frac{1}{2^8}[\int 3-4\cos 4w+\cos 8w-3\cos 2w\,dw+\int 4\cos 4w\cos 2w-\cos 8w\cos 2w\,dw] \tag{39}$$

$$\tag{40}$$

We know how to do the integral on the left, but for the one on the right we just need to know the integrals $\int \cos 4w \cos 2w\, dw$, $\int \cos 8w \cos 2w\, dw$.

We'll use the multiplication formula $\cos x \cos y = \frac{1}{2}(\cos(x-y)+\cos(x+y))$:

$$\cos 4w\cos 2w = \frac{1}{2}(\cos(4w-2w)+\cos(4w+2w)) = \frac{1}{2}(\cos 2w+\cos 6w) \tag{41}$$

$$\cos 8w\cos 2w = \frac{1}{2}(\cos(8w-2w)+\cos(8w+2w)) = \frac{1}{2}(\cos 6w+\cos 10w) \tag{42}$$

Then

$$\int\cos 4w\cos 2w\,dw = \int\frac{1}{2}(\cos 2w+\cos 6w)\,dw = \frac{1}{2}[\sin(2w)/2+\sin(6w)/6]+C \tag{43}$$

$$\int\cos 8w\cos 2w\,dw = \int\frac{1}{2}(\cos 6w+\cos 10w)\,dw = \frac{1}{2}[\sin(6w)/6+\sin(10w)/10]+C. \tag{44}$$

We return to our integral:

$$I = \frac{1}{2^8}\int 3-4\cos 4w+\cos 8w-3\cos 2w+4\cos 4w\cos 2w-\cos 8w\cos 2w\,dw \tag{45}$$

$$= \frac{1}{2^8}[\int 3-4\cos 4w+\cos 8w-3\cos 2w\,dw+\int 4\cos 4w\cos 2w-\cos 8w\cos 2w\,dw] \tag{46}$$

$$= \frac{1}{2^8}(3w-\sin 4w+\sin(8w)/8-3\sin(2w)/2+2\sin(2w)/2+2\sin(6w)/3-\sin(6w)/12+\sin(10w)/20)+C \tag{47}$$

and we're finally done.